

ON RANDOM SURFACE AREA

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ABSTRACT. Consider a random smooth Gaussian field $G(x) : F \rightarrow \mathbb{R}$, where F is a compact in \mathbb{R}^d . We derive a formula for average area of a surface generated by the equation $G(x) = 0$ and give some applications. As an auxiliary result we obtain an integral expression for area of a surface induced by zeros of a *non-random* smooth field.

Keywords: random Gaussian field, surface area, Favard measure, coarea formula, Rice formula.

1. RESULTS

Consider a compact set $F \subset \mathbb{R}^d$. By ∂F denote the boundary of F . We assume that the area of ∂F is finite (the notion of area is defined below). Let $G(x) : F \rightarrow \mathbb{R}$ be a random Gaussian field. Put $m(x) = \mathbf{E} G(x)$ and $\sigma^2(x) = \text{Var } G(x)$. Here and below we assume that $\sigma(x) > 0$ for all $x \in F$ and $G \in \mathcal{C}^1(F)$ a.s. It is known that the supremum of a continuous Gaussian field defined on a compact is summable (see [10]). Therefore, by Kolmogorov's Theorem on differentiation of mathematical expectations with respect to a parameter (see [4]), we have $m, \sigma \in \mathcal{C}^1(F)$. Let G'_i, σ'_i denote partial derivatives of G, σ with respect to i th variable. By ∇ denote a gradient of a function (a vector field whose components are partial derivatives).

Consider a zero set of the field G

$$G^{-1}(0) = \{x \in F \mid G(x) = 0\}.$$

With probability one $G^{-1}(0)$ is a compact smooth $(d-1)$ -dimensional submanifold in \mathbb{R}^d , i.e., a compact smooth surface.

The problem we are interested in is a calculation of average area of the surface $G^{-1}(0)$. Substituting G/σ for G does not change $G^{-1}(0)$. Therefore we may assume that $\sigma \equiv 1$. We prove that

$$(1) \quad \mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{1}{\sqrt{2\pi}} \int_F e^{-m^2(x)/2} \mathbf{E} \|\nabla G(x)\| dx.$$

For this purpose we derive an auxiliary formula for area of a surface generated by zeros of a non-random smooth field $g(x) : F \rightarrow \mathbb{R}$:

$$(2) \quad \lambda_{d-1}[g^{-1}(0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_F \cos[ug(x)] \|\nabla g(x)\| dx.$$

Before we proceed with the exact results formulation, we need to define the notion of area. There exist several well-known definitions of area of a $(d-1)$ -dimensional submanifold in \mathbb{R}^d : a surface Lebesgue measure, a Hausdorff measure, a Favard measure. In general they are not equivalent. However in case of compact \mathcal{C}^1 -smooth manifolds all three definitions coincide. Therefore we may choose any one. To prove (2) the best choice for λ_{d-1} is a Favard measure (for exact definition see Sect. 3). If $d = 1$, then by $\lambda_0(A)$ we denote the cardinality of a set A (may be infinite).

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Recall that F is supposed to be compact and $\lambda_{d-1}[\partial F] < \infty$.

Theorem 1. Suppose $g \in \mathcal{C}^1(F)$ and

- (a) $\lambda_{d-1}[(\nabla g)^{-1}(0)] < \infty$;
- (b) $\lambda_{d-1}[g^{-1}(0) \cap \partial F] = 0$.

Then (2) holds.

Remark. The proof of Theorem 1 shows that it is possible to get rid of condition (b). Then (2) becomes

$$\lambda_{d-1}[g^{-1}(0)] - \frac{1}{2}\lambda_{d-1}[g^{-1}(0) \cap \partial F] = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_F \cos[ug(x)] \|\nabla g(x)\| dx.$$

We shall not exploit this generalization at a later stage.

Theorem 2. Suppose a random field $G \in \mathcal{C}^1(F)$ a.s. and

- (a') $\mathbf{E} \lambda_{d-1}\left[\left(\nabla \frac{G}{\sigma}\right)^{-1}(0)\right] < \infty$;
- (b') $\sigma(x) > 0$ for all $x \in F$.

Then

$$(3) \quad \mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{1}{\sqrt{2\pi}} \int_F \exp\left\{-\frac{m^2(x)}{2\sigma^2(x)}\right\} \mathbf{E} \left\| \nabla \frac{G(x)}{\sigma(x)} \right\| dx.$$

The proofs of the theorems are in Sect. 4. The auxiliary lemmas are in Sect. 3. The applications of Theorem 2 are in Sect. 2.

2. APPLICATIONS OF THEOREM 2

2.1. Coarea formula.

Example 1. Suppose a function g satisfies the conditions of Theorem 1. Then

$$(4) \quad \int_{-\infty}^{\infty} \lambda_{d-1}[g^{-1}(u)] du = \int_F \|\nabla g(x)\| dx.$$

Proof. Consider $G(x) = g(x) - \xi$, where ξ is a Gaussian r.v. with $\mathbf{E} \xi = 0$ and $\mathbf{D} \xi = \sigma^2$. Then (3) becomes

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \lambda_{d-1}[g^{-1}(u)] e^{-\frac{u^2}{2\sigma^2}} du = \frac{1}{\sqrt{2\pi}} \int_F e^{-g^2(x)/(2\sigma^2)} \frac{\|\nabla g(x)\|}{\sigma} dx.$$

To obtain (4) it remains to multiply both sides by $\sqrt{2\pi\sigma^2}$ and apply the Monotone convergence theorem (as $\sigma \rightarrow \infty$). \square

Relation (4) is called “the coarea formula”. It was obtained by H. Federer in [7].

2.2. Centered Gaussian field. By \mathbb{S}^{d-1} denote a $(d-1)$ -dimensional unit sphere with a Lebesgue measure $\mu_{d-1}(ds)$.

Example 2. If $G(x)$ satisfies the conditions of Theorem 2 and $m(x) \equiv 0$, then

$$(5) \quad \mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F dx \int_{\mathbb{S}^{d-1}} \sqrt{s^\top \Sigma(x) s} \mu_{d-1}(ds),$$

where $\Sigma(x)$ is a covariation matrix of $\nabla\{G(x)/\sigma(x)\}$.

Proof. The proof is by Lemma 7 (see Sect. 3) which we apply to (3). \square

Remark. Relation (5) is easily extended to the case of $m(x) \equiv u$, $\sigma(x) \equiv 1$:

$$(6) \quad \mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F e^{-u^2/2} dx \int_{\mathbb{S}^{d-1}} \sqrt{s^\top \Sigma(x) s} \mu_{d-1}(ds).$$

Corollary. *Under the conditions of Example 2*

$$(7) \quad \mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F dx \int_{\mathbb{S}^{d-1}} \sigma^{-1} \left[\sum_{i,j=1}^d (\mathbf{E} G'_i G'_j - \sigma'_i \sigma'_j) s_i s_j \right]^{1/2} \mu_{d-1}(ds).$$

Proof. The proof follows from the fact that

$$\Sigma = \left(\frac{\mathbf{E} G'_i G'_j - \sigma'_i \sigma'_j}{\sigma^2} \right)_{i,j=1}^d.$$

□

2.3. Linear Gaussian field.

Example 3. Suppose $G(x) = \langle h(x), \xi \rangle$, where $h = (h^1, \dots, h^n)^\top : F \rightarrow \mathbb{R}^n$ is a vector function from the class $\mathcal{C}^1(F)$ and ξ is a n -dimensional centered Gaussian vector with the identity covariation matrix. Then

$$(8) \quad \mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F dx \int_{\mathbb{S}^{d-1}} \|J_h(x)s\| \mu_{d-1}(ds),$$

where J_h is the Jacobian n -by- d matrix of $h/\|h\|$.

Proof. We have $\Sigma = J_h^\top J_h$ in (5). □

Remark. If we consider a centered Gaussian vector with an arbitrary covariation matrix Λ , then $\Sigma = J_h^\top \Lambda J_h$ and (5) becomes

$$\mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F dx \int_{\mathbb{S}^{d-1}} \sqrt{s^\top J_h^\top(x) \Lambda J_h(x) s} \mu_{d-1}(ds).$$

For $d = 1$ this formula was obtained by A. Edelman and E. Kostlan in [6, Theorem 3.1].

Corollary. Suppose under the conditions of Example 3 the rank of J_h equals k . By $\sigma_1, \dots, \sigma_k$ denote the nonzero singular values of the matrix J_h , i.e., the nonnegative square roots of the eigenvalues of $J_h J_h^\top$. Then

$$\mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F dx \int_{\mathbb{S}^{d-1}} \left(\sum_{j=1}^k \sigma_j(x) s_j^2 \right)^{1/2} \mu_{d-1}(ds).$$

Proof. It is known from linear algebra (see, e.g., [5]) that the matrix J_h may be written in the singular form $J_h = VQW$, where V, W are n -by- n and d -by- d unitary matrices. The n -by- d matrix Q is diagonal. The diagonal elements are the singular values of the matrix J_h . We have

$$\|J_h s\| = \|VQW s\| = \|QW s\|.$$

To conclude the proof, it remains to apply this to (8) and make a change of variables $s' = Ws$. □

Now we derive another form of $\mathbf{E} \lambda_{d-1}[G^{-1}(0)]$ which will be useful for us later.

Example 4. Under the conditions of Example 3

$$(9) \quad \mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \times \int_F dx \int_{\mathbb{S}^{d-1}} \left(\sum_{i,j=1}^d \frac{\|h\|^2 \langle h'_i, h'_j \rangle - \langle h, h'_i \rangle \langle h, h'_j \rangle}{\|h\|^4} s_i s_j \right)^{1/2} \mu_{d-1}(ds),$$

where

$$h'_i = \left(\frac{\partial h^1}{\partial x_i}, \dots, \frac{\partial h^n}{\partial x_i} \right)^\top.$$

Proof. We have

$$\sigma = \|h\|, \quad \mathbf{E} G'_i G'_j = \langle h'_i, h'_j \rangle, \quad \sigma'_i = \|h\|^{-1} \langle h, h'_i \rangle.$$

It remains to apply (7). \square

2.4. Zeros of random polynomial.

Example 5. Consider $G(t) = \xi_n t^n + \dots + \xi_1 t + \xi_0, t \in F \subset \mathbb{R}$, where $\{\xi_i\}$ are independent standard Gaussian random variables. Then

$$\mathbf{E} \lambda_0[G^{-1}(0)] = \frac{1}{\pi} \int_F \frac{[A_n(t)C_n(t) - B_n^2(t)]^{1/2}}{A_n(t)} dt,$$

where

$$A_n(t) = \sum_{j=0}^n t^{2j}, \quad B_n(t) = \sum_{j=0}^n j t^{2j-1}, \quad C_n(t) = \sum_{j=0}^n j^2 t^{2j-2}.$$

Proof. The proof follows from (9). \square

This formula was obtained by M. Kac in [8]. He also derived the asymptotic relation

$$\mathbf{E} \lambda_0[G^{-1}(0)] = \frac{2}{\pi} \log n \cdot (1 + o(1)), \quad n \rightarrow \infty,$$

for $F = [-\infty, \infty]$.

2.5. Random algebraic surface.

Example 6. Consider $G(x) = \sum_{\alpha} \xi_{\alpha} x^{\alpha}$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index, the summation is taken over all α such that $0 \leq \alpha_j \leq n$, and ξ_{α} are independent standard Gaussian random variables. Then

$$(10) \quad \mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F dx \int_{\mathbb{S}^{d-1}} \left(\sum_{i=1}^d \frac{A_n(x_i)C_n(x_i) - B_n^2(x_i)}{A_n^2(x_i)} s_i^2 \right)^{1/2} \mu_{d-1}(ds).$$

Proof. Using the notations of Subsection 2.3, we get

$$\|h(x)\|^2 = \sum_{\alpha} x^{2\alpha} = \prod_{k=1}^d A_n(x_k),$$

$$\langle h(x), h'_i(x) \rangle = \frac{1}{2} \frac{\partial}{\partial x_i} \|h(x)\|^2 = B_n(x_i) \prod_{k \neq i} A_n(x_k)$$

and

$$\langle h'_i(x), h'_j(x) \rangle = \sum_{\alpha} \alpha_i x^{\alpha - \epsilon_i} \alpha_j x^{\alpha - \epsilon_j} = \begin{cases} B_n(x_i)B_n(x_j) \prod_{k \neq i,j} A_n(x_k) & \text{for } i \neq j, \\ C_n(x_i) \prod_{k \neq i} A_n(x_k) & \text{for } i = j, \end{cases}$$

where ϵ_i denotes the multi-index in which the i -th position is occupied by one and all the other positions are occupied by zeros. These relations imply that for $i \neq j$

$$\|h\|^2 \langle h'_i, h'_j \rangle - \langle h, h'_i \rangle \langle h, h'_j \rangle = 0$$

and for $i = j$

$$\|h\|^2 \langle h'_i, h'_j \rangle - \langle h, h'_i \rangle \langle h, h'_j \rangle = \|h\|^4 \frac{A_n(x_i)C_n(x_i) - B_n^2(x_i)}{A_n^2(x_i)}.$$

It remains to apply (9). \square

Formula (10) was obtained by I.A. Ibragimov and S.S. Podkorytov in [2]. They also derived the asymptotic relation

$$\mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\log d}{\pi} \lambda_{d-1}[F \cap \Gamma] \cdot (1 + o(1)), \quad n \rightarrow \infty,$$

where

$$\Gamma = \bigcup_{j=1}^d \{x \mid |x_j| = 1\},$$

provided that $\lambda_{d-1}[\partial F \cap \Gamma] = 0$.

2.6. Random surface of Kostlan-Shub-Smale.

Example 7. Consider $G(x) = \sum_{\alpha} \xi_{\alpha} x^{\alpha}$, where the summation is taken over all nonnegative α such that $\alpha_1 + \dots + \alpha_d \leq n$ and ξ_{α} are independent Gaussian random variables with $\mathbf{E} \xi_{\alpha} = 0$ and $\mathbf{D} \xi_{\alpha} = C_n^{\alpha}$, where

$$C_n^{\alpha} = \frac{n!}{\alpha_1! \dots \alpha_d! (n - \alpha_1 - \dots - \alpha_d)!}.$$

Then

$$\mathbf{E} \lambda_{d-1}[G^{-1}(0)] = \sqrt{n} \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F \frac{dx}{1 + \|x\|^2} \int_{\mathbb{S}^{d-1}} \sqrt{1 + \|x\|^2 - \langle x, s \rangle^2} \mu_{d-1}(ds).$$

Proof. Using the notations of Subsection 2.3, we get

$$\|h(x)\|^2 = \sum_{\alpha} C_n^{\alpha} x^{2\alpha} = \left(1 + \sum_{k=1}^d x_k^2\right)^n,$$

$$\langle h(x), h'_i(x) \rangle = \frac{1}{2} \frac{\partial}{\partial x_i} \|h(x)\|^2 = n x_i \left(1 + \sum_{k=1}^d x_k^2\right)^{n-1}.$$

For $i \neq j$

$$\begin{aligned} \langle h'_i(x), h'_j(x) \rangle &= \sum_{\alpha} C_n^{\alpha} \alpha_i x^{\alpha - \epsilon_i} \alpha_j x^{\alpha - \epsilon_j} \\ &= n(n-1) x_i x_j \sum_{\alpha} C_{n-2}^{\alpha} x^{2\alpha - 2\epsilon_i - 2\epsilon_j} = n(n-1) x_i x_j \left(1 + \sum_{k=1}^d x_k^2\right)^{n-2} \end{aligned}$$

and for $i = j$

$$\begin{aligned} \langle h'_i(x), h'_j(x) \rangle &= \sum_{\alpha} C_n^{\alpha} \alpha_i x^{\alpha - \epsilon_i} \alpha_i x^{\alpha - \epsilon_i} \\ &= \sum_{\alpha} C_n^{\alpha} \alpha_i x^{2\alpha - 2\epsilon_i} + \sum_{\alpha} C_n^{\alpha} \alpha_i (\alpha_i - 1) x^{2\alpha - 2\epsilon_i} \\ &= n \sum_{\alpha} C_{n-1}^{\alpha} x^{2\alpha - 2\epsilon_i} + n(n-1) x_i^2 \sum_{\alpha} C_{n-2}^{\alpha} x^{2\alpha - 4\epsilon_i} \\ &= n \left(1 + \sum_{k=1}^d x_k^2\right)^{n-1} + n(n-1) x_i^2 \left(1 + \sum_{k=1}^d x_k^2\right)^{n-2}. \end{aligned}$$

These relations imply that for $i \neq j$

$$\|h\|^2 \langle h'_i, h'_j \rangle - \langle h, h'_i \rangle \langle h, h'_j \rangle = -n \left(1 + \sum_{k=1}^d x_k^2\right)^{2n-2} x_i x_j$$

and for $i = j$

$$\|h\|^2 \langle h'_i, h'_j \rangle - \langle h, h'_i \rangle \langle h, h'_j \rangle = n \left(1 + \sum_{k=1}^d x_k^2 \right)^{2n-2} \left(1 + \sum_{k \neq i}^d x_k^2 \right).$$

Therefore, using (9) we get

$$\begin{aligned} \mathbf{E} \lambda_{d-1}[G^{-1}(0)] &= \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \sqrt{n} \int_F \left(1 + \sum_{k=1}^d x_k^2 \right)^{-1} dx \\ &\quad \times \int_{\mathbb{S}^{d-1}} \left(- \sum_{i \neq j}^d x_i x_j s_i s_j + \sum_{i=1}^d \left(1 + \sum_{k \neq i}^d x_k^2 \right) s_i^2 \right)^{1/2} \mu_{d-1}(ds) \\ &= \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \sqrt{n} \int_F (1 + \|x\|^2)^{-1} dx \int_{\mathbb{S}^{d-1}} \sqrt{1 + \|x\|^2 - \langle x, s \rangle^2} \mu_{d-1}(ds). \end{aligned}$$

□

Remark. *Thus,*

$$\mathbf{E} \lambda_{d-1}[G^{-1}(0)] = C_F \sqrt{n},$$

where C_F depends only on F and d . M. Shub and S. Smale obtained a similar result for the number of zeros of a system of d polynomials in [13].

Corollary. *For $d = 1$ we get*

$$\mathbf{E} \lambda_0[G^{-1}(0)] = \sqrt{n} \int_F \frac{dx}{\pi(1+x^2)}.$$

This relation was obtained by E. Kostlan in [9].

2.7. Random trigonometric surface. By $|F|$ denote a volume of F (i.e., a Lebesgue measure in \mathbb{R}^d).

Example 8. *Consider*

$$G(x) = \sum_{\alpha} [\xi_{\alpha} \cos \langle \alpha, x \rangle + \eta_{\alpha} \sin \langle \alpha, x \rangle],$$

where the summation is taken over all α such that $0 \leq \alpha_j \leq n$ and $\xi_{\alpha}, \eta_{\alpha}$ are independent standard Gaussian random variables. Then

$$\mathbf{E} \lambda_{d-1}[G^{-1}(0)] = n \frac{\Gamma(\frac{d+1}{2})}{4\pi^{(d+1)/2}} |F| \int_{\mathbb{S}^{d-1}} \left((s_1 + \dots + s_d)^2 + \frac{n+2}{3n} \right)^{1/2} \mu_{d-1}(ds).$$

Proof. Using the notations of Subsection 2.3, we get

$$\|h(x)\|^2 = (n+1)^d, \quad \langle h(x), h'_i(x) \rangle = \frac{1}{2} \frac{\partial}{\partial x_i} \|h(x)\|^2 = 0$$

and

$$\langle h'_i(x), h'_j(x) \rangle = \sum_{\alpha} \alpha_i \alpha_j = \begin{cases} (n+1)^{d-2} \left(\frac{n(n+1)}{2} \right)^2 & \text{for } i \neq j, \\ (n+1)^{d-1} \frac{n(n+1)(2n+1)}{6} & \text{for } i = j. \end{cases}$$

It remains to apply (9). □

Corollary (1).

$$\mathbf{E} \lambda_{d-1}[G^{-1}(0)] = c_d |F| n \cdot (1 + o(1)), \quad n \rightarrow \infty,$$

where c_d depends only on the dimension d .

Corollary (2). For $d = 1$ we get

$$\mathbf{E} \lambda_0[G^{-1}(0)] = \frac{1}{\pi} |F| \sqrt{\frac{n(2n+1)}{6}}.$$

This formula was obtained by C. Qualls in [11].

2.8. Level sets of homogeneous Gaussian field.

Example 9. Let $G(x)$ be a homogeneous Gaussian field with a spectral measure ν . Suppose ν satisfies the conditions of Theorem 1. For the sake of simplicity, we assume that $m(x) \equiv 0$ and $\sigma(x) \equiv 1$. Then

$$\mathbf{E} \lambda_{d-1}[G^{-1}(u)] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} |F| e^{-u^2/2} \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{R}^d} \langle s, z \rangle^2 \nu(dz) \right)^{1/2} \mu_{d-1}(ds).$$

Proof. By the spectral representation theorem,

$$\mathbf{E} G(x)G(y) = \int_{\mathbb{R}^d} e^{i\langle x-y, z \rangle} \nu(dz).$$

Differentiating this twice and putting $x = y = 0$, we get

$$\mathbf{E} G'_i(0)G'_j(0) = \int_{\mathbb{R}^d} z_i z_j \nu(dz).$$

Applying (6) to $G(x) - u$, we obtain

$$\begin{aligned} \mathbf{E} \lambda_{d-1}[G^{-1}(u)] &= \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} e^{-u^2/2} |F| \int_{\mathbb{S}^{d-1}} \left(\sum_{i,j=1}^d s_i s_j \int_{\mathbb{R}^d} z_i z_j \nu(dz) \right)^{1/2} \mu_{d-1}(ds) \\ &= \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} |F| e^{-u^2/2} \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{R}^d} \langle s, z \rangle^2 \nu(dz) \right)^{1/2} \mu_{d-1}(ds). \end{aligned}$$

□

Corollary (1). We have

$$\frac{1}{\pi} \gamma_1 e^{-u^2/2} |F| \leq \mathbf{E} \lambda_{d-1}[G^{-1}(0)] \leq \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi} \Gamma(\frac{d}{2})} \gamma_2 e^{-u^2/2} |F|,$$

where

$$\gamma_k = \left(\int_{\mathbb{R}} \|z\|^k \nu(dz) \right)^{1/k}.$$

Proof. By Jensen's inequality, Fubini's theorem and Lemma 2 (see Sect. 3), we get

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{R}^d} \langle s, z \rangle^2 \nu(dz) \right)^{1/2} \mu_{d-1}(ds) &\geq \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{\mathbb{R}^d} |\langle s, z \rangle| \nu(dz) \\ &= \int_{\mathbb{R}^d} \nu(dz) \int_{\mathbb{S}^{d-1}} |\langle s, z \rangle| \mu_{d-1}(ds) = \int_{\mathbb{R}^d} \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \|z\| \nu(dz) = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \gamma_1. \end{aligned}$$

On the other hand, it follows from the Cauchy—Schwarz inequality that $\|\langle s, z \rangle\| \leq \|s\| \|z\| = \|z\|$. Therefore,

$$\int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{R}^d} \langle s, z \rangle^2 \nu(dz) \right)^{1/2} \mu_{d-1}(ds) \leq \omega_{d-1} \left(\int_{\mathbb{R}^d} \|z\|^2 \nu(dz) \right)^{1/2} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \gamma_2.$$

□

Corollary (2). *For $d = 1$ we get*

$$\mathbf{E} \lambda_0[G^{-1}(u)] = \frac{\gamma_2}{\pi} e^{u^2/2} |F| .$$

This formula was obtained by S. O. Rice in [12].

3. AUXILIARY LEMMAS

Let us recall that to define a $(d - 1)$ -dimensional Favard measure of a set A , project it onto a $(d - 1)$ -dimensional linear hyperplane, take the Lebesgue measure (counting multiplicities), average over all such projections, and normalize properly:

$$(11) \quad \lambda_{d-1}[A] = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d-1}{2}}} \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{s^\perp} \lambda_0[\{s_y^\perp\}^\perp \cap A] dy ,$$

where s^\perp is the linear hyperplane orthogonal to the unit vector $s \in \mathbb{S}^{d-1}$ and $\{s_y^\perp\}^\perp$ is the line through $y \in s^\perp$ orthogonal to s^\perp .

Let us introduce the notations which we shall use in this section. Put

$$M = \sup_{R>0} \left| \int_{-R}^R \frac{\sin u}{u} du \right| .$$

It follows from Lemma 1 (see below) that $M < \infty$. By ω_k denote area of a k -dimensional sphere:

$$\omega_k = \frac{2\pi^{(k+1)/2}}{\Gamma(\frac{k+1}{2})} .$$

Throughout this section we assume that a function g satisfies the conditions of Theorem 1. By g'_s denote a partial derivative of g with respect to the direction $s \in \mathbb{S}^{d-1}$.

Lemma 1. *For all $t \in \mathbb{R}$*

$$(12) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin tu}{u} du = \text{sign } t .$$

Proof. See, i.e., [1]. □

Lemma 2. *For all $x \in \mathbb{R}^d$*

$$(13) \quad \int_{\mathbb{S}^{d-1}} |\langle x, s \rangle| \mu_{d-1}(ds) = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \|x\| .$$

Proof. Omit the trivial case when $x = 0$. Consider a Borel set A such that $A \subset x^\perp$ and $\lambda_{d-1}[A] = \|x\|$. Let us apply (11). It is clear that the integrand $\int_{s^\perp} \lambda_0[\{s_y^\perp\}^\perp \cap A] dy$ is equal to area of the projection of A onto the linear hyperplane s^\perp . On the other hand, if we project a set from one hyperplane to another, then area of the set multiplies by the cosine of the angle between the hyperplanes. Therefore,

$$\int_{s^\perp} \lambda_0[\{s_y^\perp\}^\perp \cap A] dy = \lambda_{d-1}[A] \cdot \left| \left\langle \frac{x}{\|x\|}, s \right\rangle \right| = |\langle x, s \rangle| .$$

Applying this to (11) and replacing $\lambda_{d-1}[A]$ by $\|x\|$, we obtain (13). □

The next lemma is due to M. Kac (see, e.g., [3]).

Lemma 3. *If $f(t)$ continuous for $a \leq t \leq b$ and continuously differentiable for $a < t < b$ has a finite number of turning points (i.e., only a finite number of points at which $f'(t)$ vanishes in (a, b)) then the number of zeros of $f(t)$ in (a, b) is given by the formula*

$$(14) \quad \lambda_0[f^{-1}(0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_a^b \cos[uf(t)] |f'(t)| dt .$$

Multiple zeros are counted once and if either a or b is a zero it counted as $1/2$.

Remark. This statement can be easily extended to the case of the union of a finite number of intervals. We shall use this form in the sequel.

Proof. For the readers convenience we present the proof from [3]. Let $\alpha_1, \dots, \alpha_k$ be the abscissas of the turning points:

$$a = \alpha_0 \leq \alpha_1 < \dots < \alpha_k \leq \alpha_{k+1} = b.$$

We have

$$\begin{aligned} \int_a^b \cos[uf(t)] |f'(t)| dt &= \sum_{j=0}^k \int_{\alpha_j}^{\alpha_{j+1}} \cos[uf(t)] |f'(t)| dt \\ &= \sum_{j=0}^k \left\{ \pm \int_{\alpha_j}^{\alpha_{j+1}} \cos[uf(t)] f'(t) dt \right\} = \sum_{j=0}^k \left\{ \pm \frac{\sin[uf(\alpha_{j+1})] - \sin[uf(\alpha_j)]}{u} \right\}, \end{aligned}$$

where the sign $+$ is attached if $f(t)$ is increasing between α_j and α_{j+1} and the sign $-$ if it is decreasing. Thus using (12) we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_a^b \cos[uf(t)] |f'(t)| dt \\ &= \sum_{j=0}^k \left\{ \pm \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin[uf(\alpha_{j+1})] - \sin[uf(\alpha_j)]}{u} du \right\} \\ &= \sum_{j=0}^k \left\{ \pm \frac{\text{sign } f(\alpha_{j+1}) - \text{sign } f(\alpha_j)}{2} \right\} = \lambda_0[f^{-1}(0)]. \end{aligned}$$

□

Lemma 4. If $f(t)$ continuous for $a \leq t \leq b$ and continuously differentiable for $a < t < b$ has k turning points, then uniformly for $R > 0$

$$\left| \int_{-R}^{+R} du \int_a^b \cos[uf(t)] |f'(t)| dt \right| \leq 2M(k+1).$$

Proof. In the same way as in Lemma 3 we have

$$\begin{aligned} \left| \int_{-R}^{+R} du \int_a^b \cos[uf(t)] |f'(t)| dt \right| \\ &= \left| \sum_{j=0}^k \left\{ \pm \int_{-R}^R \frac{\sin[uf(\alpha_{j+1})] - \sin[uf(\alpha_j)]}{u} du \right\} \right| \\ &= \left| \sum_{j=0}^k \pm \left\{ \int_{-Rf(\alpha_{j+1})}^{+Rf(\alpha_{j+1})} \frac{\sin u}{u} du - \int_{-Rf(\alpha_j)}^{+Rf(\alpha_j)} \frac{\sin u}{u} du \right\} \right| \\ &\leq 2(k+1) \sup_{t \in \mathbb{R}} \left| \int_{-t}^{+t} \frac{\sin u}{u} du \right| = 2M(k+1). \end{aligned}$$

□

Corollary. If we replace $[a, b]$ by a set H consisting of the union of l intervals, then uniformly for $R > 0$

$$(15) \quad \left| \int_{-R}^{+R} du \int_H \cos[uf(t)] |f'(t)| dt \right| \leq 2M(k+l).$$

Lemma 5. *The following inequality holds:*

$$\int_{\mathbb{S}^{d-1}} \lambda_{d-1}[g_s'^{-1}(0)] \mu_{d-1}(ds) \leq \omega_{d-1} \lambda_{d-1}[(\nabla g)^{-1}(0)] + \omega_{d-2}|F|.$$

Proof. We have

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \lambda_{d-1}[g_s'^{-1}(0)] \mu_{d-1}(ds) &= \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_F \mathbf{1}\{g'_s(y) = 0\} \lambda_{d-1}(dy) \\ &\leq \omega_{d-1} \lambda_{d-1}[(\nabla g)^{-1}(0)] + \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{F \setminus (\nabla g)^{-1}(0)} \mathbf{1}\{g'_s(y) = 0\} \lambda_{d-1}(dy). \end{aligned}$$

It remains to estimate the second summands. If $\nabla g(y) \neq 0$, then the set $S(y) = \{s \in \mathbb{S}^{d-1} \mid g'_s(y) = 0\}$ is contained in a unit hypersphere of the sphere \mathbb{S}^{d-1} orthogonal to $\nabla g(y)$. Consequently $\lambda_{d-2}[S(y)] \leq \omega_{d-2}$ and by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{F \setminus (\nabla g)^{-1}(0)} \mathbf{1}\{g'_s(y) = 0\} \lambda_{d-1}(dy) \\ = \int_{F \setminus (\nabla g)^{-1}(0)} dx \int_{\mathbb{S}^{d-1}} \mathbf{1}\{f'_s(x) = 0\} \mu_{d-2}(ds) \\ = \int_{F \setminus (\nabla g)^{-1}(0)} \lambda_{d-2}[S(y)] dx \leq \int_{F \setminus (\nabla g)^{-1}(0)} \omega_{d-2} dx = \omega_{d-2}|F|. \end{aligned}$$

□

Lemma 6. *For all $R > 0$*

(16)

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{\{\langle y, s \rangle = 0\}} dy \left| \int_{-R}^R du \int_{\{y+ts \in F\}} \cos[ug(y+ts)] |g'_t(y+ts)| dt \right| \\ \leq 2M \left(\omega_{d-1} \lambda_{d-1}[(\nabla g)^{-1}(0)] + \omega_{d-2}|F| + \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \lambda_{d-1}[\partial F] \right) \end{aligned}$$

and

$$\begin{aligned} (17) \quad & \left| \int_{-R}^R du \int_F \cos[ug(x)] \|\nabla g(x)\| dx \right| \\ & \leq \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d-1}{2}}} M \left(\omega_{d-1} \lambda_{d-1}[(\nabla f)^{-1}(0)] + \omega_{d-2}|F| + \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \lambda_{d-1}[\partial F] \right). \end{aligned}$$

Proof. By $k(s, y)$ denote the number of zeros of $g'_t(y+ts)$ (may be infinite) in the set $\{t \mid y+ts \in F\}$ and by $l(s, y)$ denote the number of intervals of this set (if the set is not the union of a finite number of intervals, then we put $l(s, y) = \infty$). It follows from (15) that

$$(18) \quad \left| \int_R^R du \int_{\{t \mid y+ts \in F\}} \cos[ug(y+ts)] |g'_t(y+ts)| dt \right| \leq 2M \left(k(s, y) + l(s, y) \right).$$

If we project the set $g_s'^{-1}(0)$ onto the hyperplane $\{y \mid \langle y, s \rangle = 0\}$, then $k(s, y)$ is equal to the multiplicity of the projection at the point y . A measure does not increase under the action of projection, therefore

$$\int_{\{\langle y, s \rangle = 0\}} k(s, y) dy \leq \lambda_{d-1}[g_s'^{-1}(0)],$$

which together with Lemma 5 implies

$$(19) \quad \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{\{\langle y, s \rangle = 0\}} k(s, y) dy \leq \omega_{d-1} \lambda_{d-1}[(\nabla g)^{-1}(0)] + \omega_{d-2}|F|.$$

Further, applying the definition of a Favard measure to the boundary of F , we get

$$(20) \quad \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{\{\langle y, s \rangle = 0\}} 2l(s, y) dy = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \lambda_{d-1}[\partial F].$$

Combining (18), (19) and (20) we obtain (16).

Let us prove (17). It follows from (13) that

$$\|\nabla g(x)\| = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} |\langle \nabla g(x), s \rangle| \mu_{d-1}(ds).$$

Consequently, using Fubini's Theorem we get

$$\begin{aligned} & \left| \int_{-R}^R du \int_F \cos[ug(x)] \|\nabla g(x)\| dx \right| = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d-1)/2}} \\ & \quad \times \left| \int_{-R}^R du \int_F dx \int_{\mathbb{S}^{d-1}} \cos[ug(x)] |\langle \nabla g(x), s \rangle| \mu_{d-1}(ds) \right| = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d-1)/2}} \\ & \quad \times \left| \int_{Sd} \mu_{d-1}(ds) \int_{\{\langle y, s \rangle = 0\}} dy \int_{-R}^R du \int_{\{x+ts \in F\}} \cos[ug(y+ts)] |g'_t(y+ts)| dt \right|. \end{aligned}$$

To complete the proof it remains to apply (16). \square

Lemma 7. *Consider an n -dimensional centered Gaussian vector ξ with a covariance matrix Σ . Then*

$$\mathbf{E} \|\xi\| = \frac{\Gamma(\frac{d+1}{2})}{\sqrt{2}\pi^{d/2}} \int_{\mathbb{S}^{d-1}} \sqrt{s\Sigma s^\top} \mu_{d-1}(ds).$$

Proof. It follows from (13) and Fubini's theorem that

$$\mathbf{E} \|\xi\| = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} \mathbf{E} |\langle \xi, s \rangle| \mu_{d-1}(ds).$$

Moreover,

$$\mathbf{E} |\langle \xi, s \rangle| = \mathbf{E} |\mathcal{N}(0, 1)| \sqrt{\mathbf{D} \langle \xi, s \rangle} = \left(\frac{2}{\pi}\right)^{1/2} \sqrt{s\Sigma s^\top},$$

which completes the proof. \square

4. PROOFS OF THEOREMS

Proof of Theorem 1. Using (11) and Lemma 3, we get

$$\begin{aligned} \lambda_{d-1}[g^{-1}(0)] &= \frac{\Gamma(\frac{d+1}{2})}{4\pi^{(d+1)/2}} \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{\{\langle y, s \rangle = 0\}} dy \\ & \quad \times \int_{-\infty}^{\infty} du \int_{\{y+ts \in F\}} \cos[ug(y+ts)] |g'_t(y+ts)| dt \\ &= \frac{\Gamma(\frac{d+1}{2})}{4\pi^{(d+1)/2}} \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{\{\langle y, s \rangle = 0\}} dy \\ & \quad \times \lim_{R \rightarrow \infty} \int_{-R}^R du \int_{\{y+ts \in F\}} \cos[ug(y+ts)] |g'_t(y+ts)| dt. \end{aligned}$$

It follows from the choice of F , condition (b), and (16) that we may apply Lebesgue's theorem:

$$\begin{aligned} \lambda_{d-1}(g^{-1}(0)) &= \frac{\Gamma(\frac{d+1}{2})}{4\pi^{(d+1)/2}} \lim_{R \rightarrow \infty} \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{\{(x,s)=0\}} dy \\ &\quad \times \int_{-R}^R du \int_{\{x+ts \in F\}} \cos[ug(x+ts)] |g'_t(x+ts)| dt. \end{aligned}$$

All the domains of integration are of finite measure and the integrands are bounded. Therefore we may apply Fubini's Theorem:

$$\begin{aligned} \lambda_{d-1}(g^{-1}(0)) &= \frac{\Gamma(\frac{d+1}{2})}{4\pi^{(d+1)/2}} \lim_{R \rightarrow \infty} \int_{-R}^R du \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_{\{(x,s)=0\}} dy \\ &\quad \times \int_{\{x+ts \in F\}} \cos[ug(x+ts)] |g'_t(x+ts)| dt \\ &= \frac{\Gamma(\frac{d+1}{2})}{4\pi^{(d+1)/2}} \lim_{R \rightarrow \infty} \int_{-R}^R du \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_F \cos[ug(x)] |\langle \nabla g(x), s \rangle| dx \\ &= \frac{\Gamma(\frac{d+1}{2})}{4\pi^{(d+1)/2}} \int_{-\infty}^{\infty} du \int_{\mathbb{S}^{d-1}} \mu_{d-1}(ds) \int_F \cos[ug(x)] |\langle \nabla g(x), s \rangle| dx. \end{aligned}$$

To complete the proof it remains to apply Lemma 2. □

Let us proceed to the proof of the second theorem.

Proof of Theorem 2. To apply Theorem 1 we have to show that G satisfies conditions (a), (b) almost surely. It easily follows from (a') that (a) holds almost surely. Further, using (b'), Fubini's theorem, and $\lambda_{d-1}[\partial F] < \infty$, we obtain

$$\mathbf{E} \lambda_{d-1}[G^{-1}(0) \cap \partial F] = \mathbf{E} \int_{\partial F} \mathbf{1}\{G(y) = 0\} d\lambda_{d-1}(y) = \int_{\partial F} \mathbf{P}\{G(y) = 0\} d\lambda_{d-1}(y) = 0,$$

which implies that (b) holds a.s.

First let us prove the theorem for the case when $\sigma \equiv 1$. From (2) we get

$$\begin{aligned} \mathbf{E} \lambda_{d-1}(G^{-1}(0)) &= \mathbf{E} \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_F \cos[uG(x)] \|\nabla G(x)\| dx \\ &= \frac{1}{2\pi} \mathbf{E} \lim_{R \rightarrow \infty} \int_{-R}^R du \int_F \cos[uG(x)] \|\nabla G(x)\| dx. \end{aligned}$$

It follows from the choice of F , condition (a'), and (17) that we may apply Lebesgue's theorem:

$$\begin{aligned} \mathbf{E} \lambda_{d-1}(G^{-1}(0)) &= \frac{1}{2\pi} \lim_{R \rightarrow \infty} \mathbf{E} \int_{-R}^R du \int_F \cos[uG(x)] \|\nabla G(x)\| dx \\ &= \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_F dx \int_{-R}^R \mathbf{E} \left\{ \cos[uG(x)] \|\nabla G(x)\| \right\} du. \end{aligned}$$

We may use Fubini's Theorem in the last equality on account of

$$|\cos[uG(x)]| \|\nabla G(x)\| \leq \|\nabla G(x)\| \leq \sum_{j=1}^d |G'_j(x)|$$

and

$$\mathbf{E} \int_{-R}^R du \int_F \sum_{j=1}^d |G'_j(x)| dx \leq 2R|F| \sum_{j=1}^d \mathbf{E} \sup_{x \in F} |G'_j(x)| < \infty.$$

The right-hand side is finite because the supremum of a continuous Gaussian field defined on a compact is summable (see [10]).

Differentiating $\sigma^2 \equiv 1$, we get

$$\frac{\partial(\mathbf{E} G^2)}{\partial x_i} - 2\mathbf{E} G \frac{\partial(\mathbf{E} G)}{\partial x_i} = 0 .$$

Therefore, by Kolmogorov's Theorem on differentiation of mathematical expectations with respect to a parameter (see [4]), we have

$$\mathbf{E} G G'_i = \frac{1}{2} \mathbf{E} \frac{\partial G^2}{\partial x_i} = \frac{1}{2} \frac{\partial(\mathbf{E} G^2)}{\partial x_i} = \mathbf{E} G \frac{\partial(\mathbf{E} G)}{\partial x_i} = \mathbf{E} G \mathbf{E} G'_i .$$

In other words, G does not correlate with the components of the vector ∇G which is equivalent to the independence in the Gaussian case. Thus,

$$\begin{aligned} \mathbf{E} \left\{ \cos[uG(x)] \|\nabla G(x)\| \right\} &= \mathbf{E} \cos[uG(x)] \mathbf{E} \|\nabla G(x)\| = \mathbf{Re} \left\{ \mathbf{E} e^{iuG(x)} \right\} \mathbf{E} \|\nabla G(x)\| \\ &= \mathbf{Re} \left\{ e^{i\mathbf{u}m(x)-u^2/2} \right\} \mathbf{E} \|\nabla G(x)\| = \cos[\mathbf{u}m(x)] e^{-u^2/2} \mathbf{E} \|\nabla G(x)\| , \end{aligned}$$

which implies

$$\mathbf{E} \lambda_{d-1}(G^{-1}(0)) = \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_F \mathbf{E} \|\nabla G(x)\| dx \int_{-R}^R \cos[\mathbf{u}m(x)] e^{-u^2/2} du .$$

Using Lebesgue's Theorem and the formula

$$\int_{-\infty}^{\infty} \cos[\mathbf{u}m(x)] e^{-u^2/2} du = \sqrt{2\pi} \mathbf{Re} \left\{ \mathbf{E} e^{i\mathbf{u}m(x)\mathcal{N}(0,1)} \right\} = \sqrt{2\pi} e^{-m^2(x)/2} ,$$

we obtain

$$\begin{aligned} (21) \quad \mathbf{E} \lambda_{d-1}(G^{-1}(0)) &= \frac{1}{2\pi} \int_F \mathbf{E} \|\nabla G(x)\| dx \lim_{R \rightarrow \infty} \int_{-R}^R \cos[\mathbf{u}m(x)] e^{-u^2/2} du \\ &= \frac{1}{\sqrt{2\pi}} \int_F e^{-m^2(x)/2} \mathbf{E} \|\nabla G(x)\| dx . \end{aligned}$$

We have proved the theorem for the case when $\sigma \equiv 1$. To treat the general one consider the field G/σ . It has unit variance and its zero set coincides with the zero set of G . Thus to complete the proof it remains to apply (21) to G/σ . \square

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